Hodge structures on cohomology algebras and geometry

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Institut de mathématiques de Jussieu, CNRS,UMR 7586

0 Introduction

It is well-known (see eg [22]) that the topology of a compact Kähler manifold X is strongly restricted by Hodge theory. In fact, Hodge theory provides two sets of data on the cohomology of a compact Kähler manifold. The first data are the Hodge decompositions on the cohomology spaces $H^k(X,\mathbb{C})$ (see (1.1) where $V = H^k(X,\mathbb{Q})$); they depend only on the complex structure.

The second data, known as the Lefschetz isomorphism and the Lefschetz decomposition on cohomology (see (1.5) with $A^k_{\mathbb{R}} = H^k(X,\mathbb{R})$) depend only on the choice of a Kähler class, but remain satisfied by any symplectic class close to a Kähler class.

Both are combined to give the so-called Lefschetz bilinear relations, which lead for example to the Hodge index theorem (cf [22], 6.3.2) which computes the signature of the intersection form on the middle cohomology of an even dimensional compact Kähler manifold as an alternate sum of its Hodge numbers.

If we want to extract topological restrictions using these informations, we are faced to the following problem: neither the complex structure (or even the Hodge numbers), nor the (deformation class) of the symplectic structure (or even the symplectic class) are topological.

For this reason, only a very small number of purely topological restrictions have been extracted so far from these data. (Note however that the formality theorem [7], which uses more than the data above, is a topological statement. Similarly, non-abelian Hodge theory has provided strong restrictions on $\pi_1(X)$ (cf [3]).) The classically known restrictions are the following:

1. Due to the Hodge decomposition and the Hodge symmetry (1.2), the odd Betti numbers $b_{2i+1}(X)$ have to be even.

2. Due to the Lefschetz property, the even Betti numbers $b_{2i}(X)$ are increasing in the range $2i \leq n = dim_{\mathbb{C}} X$ and similarly the odd Betti numbers $b_{2i+1}(X)$ are increasing in the range $2i + 1 \leq n = dim_{\mathbb{C}} X$.

(Note that, as the dimension of the manifold is even, the Lefschetz property also implies the condition of evenness of odd Betti numbers.)

The purpose of this paper is to extract from the Hodge decomposition and the Lefschetz property a number of purely topological restrictions on the *cohomology algebra* of a compact Kähler manifold. We will show that these restrictions are effective even in the category of compact symplectic manifold satisfying the Lefschetz property.

In [19], [15], examples of compact symplectic manifolds which are topologically non Kähler were constructed. The examples did not have their odd Betti numbers

$b_{2i+1}(X)$ even. In [10] and [2], one can find examples of compact symplectic manifolds whose odd Betti numbers are even, but for which the Lefschetz property is satisfied by no degree 2 cohomology class.

Here, many of the examples we construct satisfy the Lefschetz property. Furthermore, they are all built starting from compact Kähler manifolds, and considering either symplectic blow-up of them in a "wrong" symplectic embedding, or complex projective bundles on them. For all of them we conclude that their rational (and even sometimes real) cohomology algebra does not satisfy the restrictions imposed to the cohomology algebra of a compact Kähler manifold.

As in [20], the key point here is the observation that the Hodge decomposition on the cohomology of a compact Kähler manifold is compatible with the cup-product (see (1.4)). This leads to a number of algebraic restrictions on the cohomology algebra of X. Note that what we provide here is only a sample of them, where we tried to separate restrictions of three kinds:

1. Restrictions on the real cohomology algebra coming from the Hodge decomposition.

2. In the spirit of [20], more subtle restrictions on the rational cohomology algebra coming from the Hodge decomposition.

3. Restrictions coming from the polarization on the Hodge structure.

These restrictions, together with examples showing that they are all effective even in the symplectic category, are described in section 3. In section 2, we give two stability results concerning cohomology algebras endowed with a polarized Hodge structure. The first one (Theorem 2.1) concerns tensor products of cohomology algebras (corresponding to taking products of manifolds):

Theorem 0.1 *Assume there is a polarized Hodge structure on a (rational or real) cohomology algebra M, and assume that*

$$M \cong A \otimes B,$$

where A and B are (rational or real) cohomology algebras. If either $A^1 = 0$ or $B^1 = 0$, then A and B are of even dimension and there are polarized Hodge structures on A and B, inducing that of M.

The other one (Theorem 2.8) concerns projective bundles :

Theorem 0.2 *Let X be a compact connected smooth oriented manifold, and let E be a complex vector bundle on X with trivial determinant. Assume that the cohomology of X is generated in degrees 1 and 2. Then if the cohomology algebra $H^*(\mathbb{P}(E),\mathbb{Q})$ carries a Hodge structure, the cohomology algebra $H^*(X,\mathbb{Q}) \subset H^*(\mathbb{P}(E),\mathbb{Q})$ has an induced Hodge structure, for which the Chern classes $c_i(E)$ are Hodge classes. A similar result holds with $H^*(\mathbb{P}(E),\mathbb{Q}), H^*(X,\mathbb{Q})$ replaced by $H^*(\mathbb{P}(E),\mathbb{R})$, $H^*(X,\mathbb{R})$.*

These results are used in section 3 to construct compact symplectic manifolds with non-Kähler rational cohomology algebras, but satisfying the Lefschetz property. In one of our examples, the criterion we use, namely the existence of a Hodge structure on the cohomology algebra, needs the *rational* cohomology algebra, while on the first two examples, the real cohomology algebra suffices to exclude the existence of a Hodge structure.

In the last section, we show that there are in fact supplementary constraints on the cohomology algebra of a compact Kähler manifold. Indeed, we finally construct an example of a compact symplectic manifold which has the *real* cohomology algebra of a Kähler manifold, and whose cohomology algebra carries a rational Hodge structure, but whose rational cohomology algebra is not the rational cohomology algebra of a compact Kähler manifold. Thus all the restrictions we met before are satisfied, but still we find supplementary restrictions of a more subtle nature, related to the result in [23]: some Hodge classes on certain compact Kähler manifolds cannot be constructed as Chern classes of holomorphic vector bundles or even analytic coherent sheaves.

1 Real and rational Hodge structures on cohomology algebras

Let us recall the notion of a Hodge structure [22], 7.1.1.

Definition 1.1 *A rational (resp. real) Hodge structure of weight k is a rational (resp. real) vector space V, together with a decomposition into a direct sum of complex vector subspaces*

$$V_{\mathbb{C}} := V \otimes \mathbb{C} = \oplus_{p+q=k} V^{p,q}, \tag{1.1}$$

satisfying the Hodge symmetry condition

$$\overline{V^{p,q}} = V^{q,p}. \tag{1.2}$$

Here the tensor product is taken over $\mathbb{Q}$ in the rational case, and over $\mathbb{R}$ in the second case.

Remark 1.2 A very classical observation is the fact that for odd k, the existence of a Hodge structure of weight k on V forces the dimension of V to be even, by the Hodge symmetry (1.2).

As is well known, the data of a Hodge structure of weight k on V is equivalent to the data of an action of $\mathbb{C}^*$ on $V_{\mathbb{R}}$, satisfying the property that $\lambda \in \mathbb{R}^*$ acts by multiplication by λ^k. Indeed, we let $z \in \mathbb{C}^*$ act on $V_{\mathbb{C}}$ by multiplication by $z^p\overline{z}^q$ on $V^{p,q}$ and the Hodge symmetry (1.2) implies that this action leaves $V_{\mathbb{R}} := V \otimes \mathbb{R}$ stable.

In this paper, we will consider what we will call rational (resp. real) cohomology algebras, that is finite dimensional graded associative $\mathbb{Q}$-algebras (resp. $\mathbb{R}$-algebras) with unit, satisfying the conditions making them good candidates to be cohomology algebras of connected compact oriented manifolds:

1. The product is graded commutative.
2. $A^i = 0$ for $i < 0$ and the term A^0 is generated over $\mathbb{Q}$ (resp. $\mathbb{R}$) by 1_A.

3. For a certain integer m that we will call the dimension of A, we have

$$A^m \cong \mathbb{Q}, \text{ resp. } A^m \cong \mathbb{R},$$

and the pairing

$$A^k \otimes A^{m-k} \to A^m \tag{1.3}$$

is perfect, for any k (so, in particular $A^k = 0$ for $k > m$).

Note that condition 2 reflects the connectivity of X, when $A = H^*(X, \mathbb{Q})$, where X is a topological space, and that the dimension of A is the dimension of X if X is a compact oriented manifold.

We will denote $\cup$ the product on A, whether A comes from geometry or not.

Definition 1.3 *A Hodge structure on a cohomology algebra A is the data of a Hodge structure of weight k on each graded piece A^k, satisfying the following compatibility property with the product of A:*

$$A^{p,q} \cup A^{p',q'} \subset A^{p+p',q+q'}. \tag{1.4}$$

Remark 1.4 If there exists a Hodge structure on a cohomology algebra A, then the dimension of A is even. Indeed, the top degree cohomology of A is endowed with a Hodge structure of weight $l = dim\, A$. On the other hand, it is of rank 1 by condition 3. By remark 1.2, it follows that l is even.

The main result of Hodge theory applied to Kähler geometry is the following ([22], 6.1.3):

Theorem 1.5 *Let X be a compact Kähler manifold. Then the Hodge decomposition on each $H^k(X, \mathbb{Q})$ equips the cohomology algebra $H^*(X, \mathbb{Q})$ with a Hodge structure.*

Our goal in this paper is to explore the *topological* restrictions deduced from the existence of a Hodge structure on $H^*(X, \mathbb{Q})$ or $H^*(X, \mathbb{R})$. We will show that these restrictions are effective even in the category of compact symplectic manifolds satisfying the Lefschetz property (see next section). It turns out that stronger restrictions are obtained using the notion of polarized Hodge structure.

1.1 Polarizations

Another important property satisfied by the cohomology algebra of a compact Kähler manifold X is the fact that the Hodge structure on it can be polarized using a Kähler class $\omega \in H^{1,1}(X)_{\mathbb{R}}$. Let us define a polarization on a cohomology algebra endowed with a Hodge structure A. The class of the polarization should be an element

$$\omega \in A^{1,1}_{\mathbb{R}} := A^{1,1} \cap A^2_{\mathbb{R}}.$$

This element, seen as an element of A^2, should satisfy the Lefschetz property: let $dim\, A = 2n$. Then for any integer k, $0 \le k \le n$, the cup-product by ω^{n-k}

$$\cup \omega^{n-k} : A^k_{\mathbb{R}} \to A^{2n-k}_{\mathbb{R}}$$

should be an isomorphism. Note that both sides have the same dimension by the duality (1.3).

Remark 1.6 The morphism $\cup\omega^{n-k}$ is anti-self-adjoint for k odd, and self-adjoint for k even, with respect to the duality $A^k_{\mathbb{R}} \cong (A^{2n-k}_{\mathbb{R}})^*$. Thus, the existence of a degree 2 class ω satisfying the Lefschetz property above implies that the dimension of $A^k_{\mathbb{R}}$ is even for odd k.

The Lefschetz property implies the Lefschetz decomposition (1.5) below (cf [22], 6.2.3): Let ω be a class satisfying the Lefschetz property above, and define for $k \leq n$ the primitive part $A^k_{\mathbb{R},prim}$ by

$$A^k_{\mathbb{R},prim} := Ker\,(\cup\omega^{n+1-k} : A^k_{\mathbb{R}} \to A^{2n-k+2}_{\mathbb{R}}).$$

Then we have for $k \leq n$

$$\bigoplus_{k-2i\geq 0} A^{k-2i}_{\mathbb{R},prim} \overset{\sum_i \cup\omega^i}{\cong} A^k_{\mathbb{R}}. \tag{1.5}$$

Observe that the $A^k_{\mathbb{R},prim}$ are real sub-Hodge structures of $A^k_{\mathbb{R}}$, which means that the corresponding complex vector spaces

$$A^k_{\mathbb{C},prim} \subset A^k_{\mathbb{C}}$$

are stable under the Hodge decomposition. This follows from condition (1.4) above, and from the fact that ω is of type $(1,1)$. If furthermore A is a rational cohomology algebra and we can choose ω to be rational, then $A^k_{\mathbb{R},prim}$ is in fact defined over $\mathbb{Q}$ and provides a rational sub-Hodge structure of A^k.

To conclude, let us mention the Riemann bilinear relations, which will play a role in section 3.3. A being as above a cohomology algebra of dimension $2n$ endowed with a Hodge structure and a class ω of type $(1,1)$ satisfying Lefschetz property, we can construct for each $k \leq n$, using the duality on A given by the generator ω^n of $A^{2n}_{\mathbb{R}}$, a non degenerate intersection pairing q_ω on $A^k_{\mathbb{R}}$, which is symmetric if k is even and alternate if k is odd, defined by:

$$q_\omega(\alpha,\beta) = \omega^{n-k} \cup \alpha \cup \beta \in A^{2n}_{\mathbb{R}} \cong \mathbb{R}.$$

(The last isomorphism here is defined up to a multiplicative coefficient).

It follows easily from the definition of the primitive parts $A^i_{\mathbb{R},prim} \subset A^i_{\mathbb{R}}$ that the Lefschetz decomposition is orthogonal for the pairing q_ω. Let us now introduce the Hermitian pairing on $A^k_{\mathbb{C}}$:

$$h_\omega(\alpha,\beta) = \iota^k q_\omega(\alpha,\overline{\beta}).$$

For bidegree reasons, using the condition (1.4), we also find that the Hodge decomposition is orthogonal with respect to the pairing h_ω.

$$h_\omega(\alpha,\beta) = 0,\ \alpha \in A^{p,q}_{\mathbb{C}},\ \beta \in A^{p',q'}_{\mathbb{C}},\ (p,q) \neq (p',q'). \tag{1.6}$$

Finally the second Riemann bilinear relations are restrictions on the signs of the Hermitian pairing h_ω restricted to the part

$$A^{p,q}_{\mathbb{C},prim} \subset A^{p,q}_{\mathbb{C}},\ p+q=k$$

defined as $A^{p,q}_{\mathbb{C},prim} = A^{p,q}_{\mathbb{C}} \cap A^k_{\mathbb{C},prim}$. If A is the cohomology algebra of a compact Kähler manifold, these restrictions are described in the following theorem.

Theorem 1.7 *Let X be a compact Kähler manifold with Kähler class ω. Then the Hermitian form h_ω is definite of sign $(-1)^{\frac{k(k-1)}{2}}\iota^{p-q-k}$ on the component*

$$\omega^r \cup H^{p,q}(X,\mathbb{C})_{prim},\ 2r+p+q=k$$

of $H^k(X,\mathbb{C})$.

A polarized cohomology algebra (A,ω) should satisfy also these sign restrictions.

At this point, we find another constraint on the Betti numbers of a compact Kähler manifold, which shows that we cannot consider separately the odd and even Betti numbers to address the question asked by Simpson in [18], namely what can be the Betti or Hodge numbers of compact Kähler manifolds. Indeed, we have the following lemma:

Lemma 1.8 *Let M be a cohomology algebra which carries a polarized Hodge structure. Let $2n = dim\, M$, and let i be an integer such that*

$$4i+2\leq n.$$

Then if $M^{2i+1}\neq 0$, we have $rk\, M^{4i+2}\geq 2$ and more precisely $rk\, M^{2i+1,2i+1}\geq 2$.

Proof. Indeed, note that some $M^{p,q}\neq\{0\}$ for some $p>q$, $p+q=2i+1$. Then there is an $\alpha\in M^{p,q}_{\mathbb{C}}$ such that $\alpha\cup\overline{\alpha}\neq 0$, otherwise this contradicts the fact that for a polarisation given by $\omega\in M^{1,1}$, the Hermitian form

$$h_\omega(\alpha,\overline{\alpha})=\iota\omega^{n-2i-1}\cup\alpha\cup\overline{\alpha}$$

is non degenerate on $M^{p,q}_{\mathbb{C}}$ by the Lefschetz decomposition (1.5) and the Riemann bilinear relations (1.7).

Hence $M^{4i+2}_{\mathbb{C}}$ contains a non zero class of type $(2i+1,2i+1)$, of the form $\beta=\alpha\cup\overline{\alpha}$. On the other hand, $M^{2i+1,2i+1}_{\mathbb{C}}$ contains ω^{2i+1}. But these two cohomology classes cannot be proportional because $\beta^2=0$, while $\omega^{4i+2}\neq 0$ by Lefschetz property and because $4i+2\leq n$. ∎

1.2 Sub-Hodge structures and a lemma of Deligne

The following lemma, communicated to the author by Deligne [6], and very much used in [20], [21], allows to detect sub-Hodge structures in a given cohomology algebra endowed with a Hodge structure. Let $A^*=\oplus_k A^k$ be a rational (resp. real) cohomology algebra endowed with a Hodge structure. Let $A^*_{\mathbb{C}}:=A^*\otimes\mathbb{C}$. Let $Z\subset A^k_{\mathbb{C}}$ be an algebraic subset which is defined by homogeneous equations expressed only using the ring structure on A^*. The examples we shall consider in this paper will often be of the form :

$$Z=\{\alpha\in A^k_{\mathbb{C}}/\alpha^l=0 \text{ in } A^{kl}_{\mathbb{C}}\},$$

where l is a given integer.

Lemma 1.9 *Let Z be as above, and let Z_1 be an irreducible component of Z. Assume the $\mathbb{C}$-vector space $<Z_1>$ generated by Z_1 is defined over $\mathbb{Q}$, (resp. over $\mathbb{R}$), that is $<Z_1>=B^k_{\mathbb{Q}}\otimes\mathbb{C}$ for some $B^k_{\mathbb{Q}}\subset A^k_{\mathbb{Q}}$ (resp. $<Z_1>=B^k_{\mathbb{R}}\otimes\mathbb{C}$ for some $B^k_{\mathbb{R}}\subset A^k_{\mathbb{R}}$). Then $B^k_{\mathbb{Q}}$ (resp. $B^k_{\mathbb{R}}$) is a rational (resp. real) sub-Hodge structure of $A^k_{\mathbb{Q}}$ (resp. $A^k_{\mathbb{R}}$).*

We refer to [20] for the proof of this lemma.

2 Stability results

2.1 Products

We prove in this section the following result:

Theorem 2.1 *Assume there is a polarized Hodge structure on a connected (rational or real) cohomology algebra M, and assume that*

$$M \cong A \otimes B,$$

where A and B are (rational or real) cohomology algebras. If either $A^1 = 0$ or $B^1 = 0$, then A and B are of even dimension and there are polarized Hodge structures on A and B, inducing that of M.

Geometrically, this implies that if a product $X \times Y$, where X and Y are smooth compact oriented manifolds, has the cohomology algebra of a Kähler compact manifold, and one of them has $b_1 = 0$, then the cohomology algebras of X and Y carry polarized Hodge structures, and thus in particular inherit all the constraints described in next section.

Another geometric consequence is the fact that if a Kähler compact manifold M is homeomorphic to a product $X \times Y$, where either $b_1(X) = 0$ or $b_1(Y) = 0$, then any polarized Hodge structure on $H^*(M)$ comes from polarized Hodge structures on $H^*(X)$ and $H^*(Y)$, a result which can be compared to a result in deformation theory: if a compact complex manifold is a product $X \times Y$, and $b_1(X) = 0$, $b_1(Y) = 0$, then small deformations of $X \times Y$ are of the form $X' \times Y'$, where X' is a deformation of X and Y' is a deformation of Y. (This is not true if only one of the assumptions $b_1(X) = 0$, $b_1(Y) = 0$ is satisfied, as a product may deform to a non trivial fiber bundle.)

Remark 2.2 The assumption that $A^1 = 0$ or $B^1 = 0$ is obviously necessary in all the statements above, as the case of a complex torus shows. Indeed, the complex torus is a product of an even number $2n$ of copies of S^1's, and thus can be written as a product of X, a product of an odd number $2d - 1$ of copies of S^1's and Y, a product of an odd number $2d' + 1$ of copies of S^1's, with $d + d' = n$.

To start the proof of the theorem, let us show the assertion of even dimensionality.

Lemma 2.3 *Under the assumptions of Theorem 2.1, A and B are even dimensional.*

Proof. Indeed, let $s = dim\, A$, $t = dim\, B$, so that

$$s + t = 2n := dim\, M.$$

As $A^1 = 0$, or $B^1 = 0$, and $M = A \otimes B$, one has

$$M^2 = A^2 \otimes B^0 \oplus A^0 \otimes B^2,$$

where we can identify canonically A^0, B^0 with $\mathbb{Q}$. Let $\omega \in M^2_{\mathbb{R}}$ be the class of a polarization. Then $\omega^n \neq 0$ in $M^{2n} = A^s \otimes B^t$.

Writing $\omega = a + b$, with $a \in A^2$ and $b \in B^2$, we conclude by writing

$$\omega^n = \oplus_{i \leq n} \binom{n}{i} a^i \otimes b^{n-i}$$

that A^s must be generated by a power of a, and B^t must be generated by a power of b. Thus s and t are even.

■

Note also that the proof showed that the algebras A and B have their top degree part generated by a power of a degree 2 element.

Let us now apply Deligne's Lemma 1.9 to get the following:

Lemma 2.4 *The subspaces*

$$A^2 \cong A^2 \otimes B^0 \subset M^2,\, B^2 \cong A^0 \otimes B^2 \subset M^2$$

are rational sub-Hodge structures of M^2.

Proof. By Deligne's lemma, it suffices to show how to recover these subspaces algebraically, using only the algebra structure of M. Let $2s = dim\, A$, $2t = dim\, B$, so that $s + t = n$. We claim that $A^2_{\mathbb{C}} \subset M^2_{\mathbb{C}}$ is an irreducible component of

$$Z \subset M^2_{\mathbb{C}},\, Z = \{m \in M^2_{\mathbb{C}},\, m^{s+1} = 0\},$$

and similarly for $B^2_{\mathbb{C}}$, with s replaced by t.

To see this, let as before $\omega = a + b$ be a decomposition of a polarizing class $\omega \in M_{\mathbb{R}}$. Then we proved in the previous lemma that $a^s \neq 0$ in A^{2s}. The tangent space to Z at the point a is described as

$$T_{Z,a} = \{m \in M^2_{\mathbb{C}},\, a^s m = 0\}.$$

Writing $m = \alpha + \beta$, $\alpha \in A^2$, $\beta \in B^2$, we conclude immediately that $\beta = 0$ for $m \in T_{Z,a}$, which shows that the Zariski tangent spaces of $A^2_{\mathbb{C}}$ and Z coincide at a. As $A^2_{\mathbb{C}} \subset Z$ is smooth, this implies that $A^2_{\mathbb{C}}$ is an irreducible component of Z. ■

Recall that a Hodge class in a rational Hodge structure M^{2k} of weight $2k$ is a rational element of M^{2k} which is also in $M^{k,k}$. As a corollary of the previous two lemmas, we get

Corollary 2.5 *The* 1*-dimensional rational spaces* $A^{2s} \subset M^{2s}$, $B^{2t} \subset M^{2t}$ *are generated by Hodge classes* η_A, η_B *of* M *of respective degrees* $2s$, $2t$.

Proof. Indeed, consider the case of A. Then, as mentioned above, A^{2s} is the image of the map

$$Sym^s A^2 \to Sym^s M^2 \to M^{2s},$$

where the first map is the inclusion, and the second is given by the product of M. As A^2 is a sub-Hodge structure of M^2, A^{2s} is also a sub-Hodge structure of M^{2s}. As it is one dimensional, it must be generated by a Hodge class. ■

We finally prove the following:

Lemma 2.6 *Let $\omega \in M^2_{\mathbb{R}}$ be a polarizing class and decompose it as*

$$\omega = a + b,\ a \in A^2_{\mathbb{R}},\ b \in B^2_{\mathbb{R}}.$$

Then the classes a and b satisfy the Lefschetz property, namely for any $k \leq s$

$$\cup a^{s-k} : A^k_{\mathbb{R}} \to A^{2s-k}_{\mathbb{R}}$$

is an isomorphism, and similarly for b and B.

Proof. Let $\alpha \in A^k_{\mathbb{R}} \subset M^k_{\mathbb{R}}$ and assume that $a^{s-k} \cup \alpha = 0$ in $A^{2s-k}_{\mathbb{R}}$. It then follows that $a^i \cup \alpha =$ in $A^{k+2i}_{\mathbb{R}}$ for all $i \geq s-k$. Let us compute now:

$$\omega^{n-k} \cup \alpha = \sum_{i \leq n-k} \binom{n-k}{i} a^i \cup b^{n-k-i} \cup \alpha.$$

As $a^i \cup \alpha = 0$ for $i \geq s-k$, the sum runs only over the pairs (i, j) with $i + j = n-k$, $i < s-k$, and thus $j > n-s = t$. As $b^{t+1} = 0$, we get $\omega^{n-k} \cup \alpha = 0$ and the Lefschetz property for (M, ω) shows that $\alpha = 0$. ■

Proof of theorem 2.1. We assert that each A^i, $B^i \subset M^i$ is a sub-Hodge structure of M^i. For $i = 1$, this follows from the fact that we have either $A^1 = 0$, $B^1 = M^1$ or $A^1 = M^1$, $B^1 = 0$ and for $i = 2$, this follows from Lemma 2.4. We choose now a polarizing class $\omega \in M^{1,1}_{\mathbb{R}}$ and decompose it into $\omega = a + b$. Then by Lemma 2.4, a and b are of type $(1, 1)$.

We prove now the result by induction on i. So assume the result is proved for $i - 1$. Then all the subspaces

$$A^l \otimes B^{l'},\ l + l' = i,\ l > 0, l' > 0$$

of M^i are sub-Hodge structures. Using now Lemma 2.4, and using a polarization $\omega = a + b \in M^{1,1}_{\mathbb{R}}$, we have that a and b are in $M^{1,1}_{\mathbb{R}}$ and thus we get that each subspace

$$a^{s-l} \cup b^{t-l'} \cup A^l_{\mathbb{R}} \otimes B^{l'}_{\mathbb{R}} \subset M^{2n-i}_{\mathbb{R}}$$

is a real sub-Hodge structure.

By lemma 2.6, this subspace is in fact equal to $A^{2s-l} \otimes B^{2t-l'} \otimes \mathbb{R}$, and thus we conclude that for $l > 0$ and $l' > 0$, $l + l' = i$

$$A^{2s-l} \otimes B^{2t-l'} \subset M^{2n-i}$$

is a rational sub-Hodge structure of M^{2n-i}, because it is rational, and tensored by $\mathbb{R}$ becomes a real sub-Hodge structure. But the orthogonal of

$$\oplus_{l>0, l'>0, l+l'=i} A^{2s-l} \otimes B^{2t-l'}$$

with respect to the intersection pairing on M is equal to $A^i \oplus B^i$. Thus $A^i \oplus B^i$ is a sub-Hodge structure of M^i. But $A^i \subset A^i \oplus B^i$ is the kernel of the restriction to $A^i \oplus B^i \subset M^i$ of the multiplication by the Hodge class $\eta_A : M^i \to M^{i+2s}$, where η_A generates A^{2s}, and similarly $B^i \subset A^i \oplus B^i$ is the kernel of the restriction to

$A^i \oplus B^i \subset M^i$ of the multiplication by the Hodge class $\eta_B : M^i \to M^{i+2t}$, where η_B generates B^{2t}. Thus we conclude that A^i and B^i are sub-Hodge structures of M^i.

To conclude the proof of the theorem, it remains to show that a polarizes the Hodge structure on $A \subset M$, and b polarizes the Hodge structure on $B \subset M$. We already proved that a and b satisfy the Lefschetz property. We have to show that the second bilinear relations (1.7) hold.

This is easy because, the same argument as in the proof of Lemma 2.6 shows that if $\alpha \in A^i_{\mathbb{R}}$, $i \leq s$ is primitive for a, namely satisfies $a^{s-i+1} \cup \alpha = 0$, then $\alpha \in M^i_{\mathbb{R}}$ is primitive for ω. Furthermore, we have, for primitive α, β, and for the choices of generators

$$\omega^n,\ a^s,\ b^t$$

of $M^{2n}_{\mathbb{R}}$, $A^{2s}_{\mathbb{R}}$, $B^{2t}_{\mathbb{R}}$ respectively:

$$\nu < \alpha, a^{s-i} \cup \beta >_A = < \alpha, \omega^{n-i} \cup \beta >_M,$$

where

$$\nu = \frac{\binom{n-i}{s-i}}{\binom{n}{s}}.$$

Thus the second bilinear relations for (M, ω) imply the second bilinear relations for (A, a). ∎

Note to conclude that, without polarizations, the same arguments prove the following result, which will be used later on:

Theorem 2.7 *Assume there is a Hodge structure on a (rational or real) cohomology algebra M, and assume that*

$$M \cong A \otimes B,$$

where A and B are (rational or real) cohomology algebras. If $A^1 = 0$, and A and B are generated in degree ≤ 2, then A and B are of even dimension and there are Hodge structures on A and B, inducing that of M.

Proof. As in the previous proof, we first use Deligne's Lemma 1.9 to prove that $A^2 \otimes 1_B$ and $1_A \otimes B^2$ are sub-Hodge structures of M^2. To prove this, note that M is generated in degree ≤ 2, which implies, as it is of even dimension $2n$, that there is a $\omega \in M^2$ such that $\omega^n \neq 0$ in M^{2n} (otherwise any monomial $\omega_1 \dots \omega_n$ with $deg\, \omega_i = 2$ would vanish in M^{2n}, contradicting the fact that M^{2n} is generated by a product of elements of degree 1 or 2). Having this, and writing $\omega = \alpha + \beta$, $\alpha \in A^2 \otimes 1_B$, $\beta \in 1_A \otimes B^2$, we conclude as before that

$$\alpha^d \neq 0 \text{ in } A^2,\ \beta^{d'} \neq 0 \text{ in } B^2,$$

for some integers d, d' such that $dim\, A = 2d$, $dim\, B = 2d'$. It follows then that $A^2_{\mathbb{C}} \otimes 1_B$ is recovered as an irreducible component of the set

$$\{m \in M^2_{\mathbb{C}},\ m^{d+1} = 0 \text{ in } M^{2d+2}_{\mathbb{C}}\},$$

and similarly for $1_A \otimes B^2_{\mathbb{C}}$. Thus by Deligne's Lemma 1.9, $A^2 \otimes 1_B$ and $1_A \otimes B^2$ are sub-Hodge structures of M^2.

As $A^1 = M^1$ the same is true in degree 1. As the algebras are generated in degree ≤ 2, it follows that $A = A \otimes 1_B$ and $B = 1_A \otimes B$ are sub-Hodge structures of M. ∎

2.2 Projective bundles

We consider now the simplest kinds of compact oriented manifolds which are close to be a product: namely bundles over a basis satisfying the Leray-Hirsch condition. Then their cohomology is additively the tensor product of the cohomology of the basis and the cohomology of the fiber, but not multiplicatively. The simplest examples of this are given by complex projective bundles $\mathbb{P}(E) \to X$ associated to complex vector bundles on compact oriented manifolds.

Theorem 2.8 *Let X be a compact smooth oriented manifold, and let E be a complex vector bundle of rank ≥ 2 on X with trivial determinant. Assume that the cohomology of X is generated in degrees 1 and 2. Then if the cohomology algebra $H^*(\mathbb{P}(E), \mathbb{Q})$ carries a Hodge structure, the cohomology algebra $H^*(X, \mathbb{Q}) \subset H^*(\mathbb{P}(E), \mathbb{Q})$ has an induced Hodge structure, for which the Chern classes $c_i(E)$ are Hodge classes. A similar result holds with $H^*(\mathbb{P}(E), \mathbb{Q}), H^*(X, \mathbb{Q})$ replaced by $H^*(\mathbb{P}(E), \mathbb{R})$, $H^*(X, \mathbb{R})$.*

Proof. Let X be of real dimension $2n$, $\pi : \mathbb{P}(E) \to X$ be the structural map. We know that

$$\Gamma := \pi^* H^2(X, \mathbb{Q}) \subset H^2(\mathbb{P}(E), \mathbb{Q})$$

is a hyperplane, which satisfies the properties that $\alpha^{n+1} = 0$, for any $\alpha \in \Gamma$. Next, the cohomology of $\mathbb{P}(E)$, as the cohomology of X, being generated in degree 1 and 2, it follows that there exists a class $\beta \in H^2(\mathbb{P}(E), \mathbb{Q})$ such that $\beta^{n+r-1} \neq 0$, where $r := rank\, E$.

Thus we conclude that the hyperplane $\Gamma_{\mathbb{C}} \subset H^2(\mathbb{P}(E), \mathbb{C})$ must be an irreducible component of the algebraic subset

$$Z := \{\beta \in H^2(\mathbb{P}(E), \mathbb{C}),\ \beta^{n+r-1} = 0\}.$$

By Deligne's Lemma 1.9, it follows that Γ is a sub-Hodge structure of $H^2(\mathbb{P}(E), \mathbb{Q})$ for the given Hodge structure. As $\pi^* : H^1(X, \mathbb{Q}) \to H^1(\mathbb{P}(E), \mathbb{Q})$ is an isomorphism, we have the same conclusion for the cohomology of degree 1. Finally, as the cohomology of X is generated in degrees 1 and 2, we conclude that the cohomology sub-algebra

$$\pi^* H^*(X, \mathbb{Q}) \subset H^*(\mathbb{P}(E), \mathbb{Q})$$

is also a sub-Hodge structure.

Observe now that the injective map

$$\pi^* : H^*(X, \mathbb{Q}) \to H^*(\mathbb{P}(E), \mathbb{Q})$$

admits as its dual map the Gysin map

$$\pi_* : H^*(\mathbb{P}(E), \mathbb{Q}) \to H^{*-2r+2}(X, \mathbb{Q})$$

which thus must be also a morphism of Hodge structures (of bidegree $(-r+1, -r+1)$ on each graded piece).

We claim that there is a class $\beta \in H^2(\mathbb{P}(E), \mathbb{Q})$, unique up to a multiplicative coefficient, such that β does not vanish modulo $\pi^* H^2(X, \mathbb{Q})$ and satisfies

$$\pi_* \beta^r = 0 \text{ in } H^2(X, \mathbb{Q}).$$

Furthermore this class must be a Hodge class. Indeed, we take for β the class $c_1(H)$ where H is the dual of the Hopf line bundle on $\mathbb{P}(E)$. Then it is a standard fact (see [9]) that

$$\pi_*\beta^r = -c_1(E).$$

As $c_1(E)$ was supposed to be 0, this proves the existence of β. As for the uniqueness, observe that for $\alpha \in H^2(X, \mathbb{Q})$,

$$\pi_*(\beta + \pi^*\alpha)^r = \pi_*\beta^r + r\alpha,$$

so that $\pi_*(\beta + \pi^*\alpha)^r = 0$ implies $\alpha = 0$.

The same argument shows that the complex line $\beta\mathbb{C} \subset H^2(\mathbb{P}(E), \mathbb{C})$ is an irreducible component of the closed algebraic subset

$$Z := \{\gamma \in H^2(\mathbb{P}(E), \mathbb{C}),\ \pi_*\gamma^r = 0\}.$$

As π_* is a morphism of Hodge structures, we conclude by Deligne's Lemma 1.9 that β must be a Hodge class.

The proof is now finished. Indeed, let β be defined (up to a multiplicative coefficient) as above. Then β satisfies in $H^*(\mathbb{P}(E), \mathbb{Q})$ a unique polynomial equation

$$\beta^r = \sum_{1 \leq i \leq r} \beta^{r-i} \cup \pi^*\alpha_i,$$

where the α_i are proportional to $c_i(E)$. As all the powers β^l's are Hodge classes on $\mathbb{P}(E)$, and the α_i's can be recovered as polynomials in the Segre classes $\sigma_j(E) = \pi_*(\beta^{r-1+j})$ (cf [9]), the α_i must be also Hodge classes on X for the induced Hodge structure on $H^*(X, \mathbb{Q})$.

∎

3 Explicit constraints

The purpose of this section is to describe explicit constraints on a cohomology algebra, imposed by the presence of a (polarized) Hodge structure. We want to separate the constraints on the *real* cohomology algebra imposed by the Hodge decomposition, which will be considered in section 3.1, from more subtle constraints related to the *rational* cohomology algebra, which will be explained in section 3.2 and from those imposed by the polarisation (section 3.3). We will illustrate the effectiveness of each of these criteria by exhibiting compact symplectic manifolds satisfying the Lefschetz property and not satisfying the considered criterion.

3.1 Constraints coming from the real Hodge structure

It is clear that constraints coming only from the real (or rational) Hodge structure, without polarizations, must involve the odd degree cohomology. Indeed, if there is no odd degree cohomology, we can put the trivial Hodge structure on all the cohomology groups, and this will obviously satisfy the compatibility conditions (1.4).

The classically known topological restriction on the cohomology algebra of a compact Kähler manifold coming from the Hodge decomposition is the fact that odd Betti numbers b_{2i+1} must be even. This condition is very restrictive for surfaces, as it is known that among compact complex surfaces, this condition characterizes the Kähler ones, a result due to Kodaira [13].

Let us refine this restriction using the cohomology algebra.

Lemma 3.1 *Let M be a real cohomology algebra. If M carries a Hodge structure, then for any pair of integers k, l with k even and l odd, the product:*

$$\mu : M^l \otimes M^k \to M^{k+l}$$

must be of even rank. More generally, if ${M'}^k \subset M^k$, ${M'}^l \subset M^l$ are sub-Hodge structures, then the product:

$$\mu : {M'}^l \otimes {M'}^k \to M^{k+l}$$

must be of even rank.

Proof. Indeed, as ${M'}^k \otimes \mathbb{C}$ and ${M'}^l \otimes \mathbb{C}$ are stable under the Hodge decomposition, that is decompose into the direct sum of their components of type (p, q), $p + q = k$, resp. $p + q = l$, it follows from the compatibility conditions (1.4) that the image $Im\, \mu_{\mathbb{C}}$ of μ, tensored by $\mathbb{C}$, is also stable under the Hodge decomposition, namely, it is the direct sum of its terms of type (p, q), $p+q = k+l$. As $k+l$ is odd, and $Im\, \mu_{\mathbb{C}}$ is defined over $\mathbb{R}$, the Hodge symmetry (1.2) is satisfied by $Im\, \mu_{\mathbb{C}}$, which implies that it is of even rank. ∎

This lemma can be combined with Deligne's Lemma 1.9, to get effective restrictions which are much stronger than the classical ones. Let us state this restriction explicitely:

Proposition 3.2 *Let M be a real cohomology algebra with Hodge structure. For some even integer $2k$, let $Z_1, \ldots, Z_r \subset M^{2k}_{\mathbb{C}}$ be algebraic subsets as in Lemma 1.9. Suppose the complex vector spaces $< Z_i >$, $i = 1, \ldots, r$ are defined over $\mathbb{R}$, that is*

$$< Z_i >= B_i \otimes \mathbb{C},\ B \subset M^{2k}.$$

Let $B := \sum_i B_i \subset M^{2k}$. Then for any odd integer l, the product map

$$B \otimes M^l \to M^{2k+l}$$

has even rank.

Indeed, Deligne's Lemma then tells us that B has to be a sub-Hodge structure, so that we can apply Lemma 3.1. ∎

Let us construct using Theorem 3.2 a compact symplectic manifold X which satisfies the Lefschetz condition, hence in particular has its odd Betti numbers even, but whose *real* cohomology algebra does not admit any Hodge structure. So in particular, this X does not have the real cohomology algebra of a compact Kähler manifold.

Example 3.3 *We start with a a complex torus $T = \mathbb{C}^3/\Gamma$ of dimension 3, and we fix a symplectic structure on T, given for example by a constant Kähler form ω on $\mathbb{C}^3$. We may even assume that T is an abelian variety and that the cohomology class*

λ_0 of the Kähler form is rational. Next, we choose 2 elements λ_1, λ_2 in $H^2(T,\mathbb{Q})$ which satisfy the property that

$$\lambda_1 \cup H^1(T,\mathbb{Q}) + \lambda_2 \cup H^1(T,\mathbb{Q})$$

has rank 11. As rank $H^1(T,\mathbb{Q}) = 6$, this means that the map

$$\mu : H^1(T,\mathbb{Q}) \oplus H^1(T,\mathbb{Q}) \to H^3(T,\mathbb{Q}), \qquad (3.7)$$
$$\mu((a,b)) = \lambda_1 \cup a + \lambda_2 \cup b$$

has a 1-dimensional kernel. Let us give an explicit example of such a pair: Choose a basis $w_1,\ldots,w_6$ of the $\mathbb{Q}$-vector space $H^1(T,\mathbb{Q}) \cong \Gamma^*_{\mathbb{Q}}$. The cohomology $H^*(T,\mathbb{Q})$ identifies to the exterior algebra $\bigwedge \Gamma^*_{\mathbb{Q}}$, the cup-product being identified with the exterior product. Let

$$\lambda_1 = w_1 \wedge w_2 + w_3 \wedge w_4,\ \lambda_2 = w_3 \wedge w_5 - w_1 \wedge w_4.$$

Then we clearly have $w_1 \wedge \lambda_1 = w_3 \wedge \lambda_2$, and it is an easy exercise to verify that the kernel of the map (3.7) is generated by this relation.

We now do the following : choosing $\epsilon \in \mathbb{Q}$ small enough, the classes $\lambda_0 + \epsilon\lambda_1$ and $\lambda_0 + \epsilon\lambda_2$ are symplectic classes, which can be represented by symplectic forms ω_1, ω_2 in the same deformation class as ω. In fact, the important point for us is the fact that $\omega_0 + \omega_1 + \omega_2$ is a symplectic form, whose class is close to $3\lambda_0$.

As the classes λ_0, λ_1, λ_2 are rational, we can find multiples $M\lambda_0$, $M\lambda_1$, $M\lambda_2$ which consist of integral symplectic classes, such that there are symplectic embeddings (we can use [12], 3.4, or approximately holomorphic embeddings (see [16] using results of [8]))

$$\phi_i : T \to \mathbb{P}^N,\ i = 1,\ 2,\ 3,$$

with $\phi_i^*\Omega = M\omega_i$, where Ω is the Fubini-Study symplectic form on $\mathbb{P}^N$.

Let $\psi : T \to \mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N$ be the map $(\phi_1,\ \phi_2,\ \phi_3)$. For the product symplectic form $\widetilde{\Omega} = p_1^*\Omega + p_2^*\Omega + p_3^*\Omega$, the image $W = \psi(T)$ is a symplectic submanifold, because $\psi^*\widetilde{\Omega} = M(\omega_0 + \omega_1 + \omega_2)$.

Our example will be the symplectic manifold X obtained as the symplectic blow-up of $\mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N$ along W. For an adequate choice of symplectic form of class $\mu_1 + \mu_2 + \mu_3 - \eta e$, with η very small, the Lefschetz property is satisfied, as it follows easily from the fact that the restriction of the symplectic class $p_1^*\omega + p_2^*\omega + p_3^*\omega$ to $W = T$ satisfies the Lefschetz property on T. (The Lefschetz property for symplectic blow-ups is studied in general in [4].)

Proposition 3.4 *The cohomology algebra of X does not satisfy the condition of Proposition 3.2, hence does not admit any real Hodge structure. In particular X does not have the cohomology algebra of a compact Kähler manifold.*

Proof. By the computation of the cohomology of a symplectic blow-up (cf [22], 7.3.3 in the complex case), we get that $H^2(X,\mathbb{Q}) = \mathbb{Q}^4$, generated by the class e of the exceptional divisor, and the classes $\mu_i := \tau^*(p_i^*[\Omega])$, $i = 1,\ 2,\ 3$, where $\tau : X \to \mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N$ is the blowing-up map. Furthermore, letting $j : E \to X$ denote the inclusion of the exceptional divisor of τ, and $\tau' : E \to W = T$ the restriction of τ, we have that $H^3(X,\mathbb{Q}) = j_*({\tau'}^* H^1(T,\mathbb{Q}))$.

Now, one sees easily that each class μ_i generates an irreducible component of the algebraic subset

$$Z = \{a \in H^2(X, \mathbb{C}),\ a^{N+1} = 0 \text{ in } H^{2N+2}(X, \mathbb{C})\}.$$

As each μ_i is rational, it must be a Hodge class by Deligne's Lemma 1.9.

Hence we proved that for any Hodge structure on $H^*(X, \mathbb{Q})$, the classes μ_i are of type $(1, 1)$. It remains to see that for two adequately chosen rational (or real) cohomology classes λ, λ' which are combinations of the μ_i, the rank of

$$\begin{aligned} \mu : H^3(X, \mathbb{Q}) \oplus H^3(X, \mathbb{Q}) &\to H^5(X, \mathbb{Q}) \\ (\alpha, \beta) &\mapsto \lambda \cup \alpha + \lambda' \cup \beta \end{aligned} \tag{3.8}$$

is odd.

Let us take

$$\lambda = \mu_2 - \frac{1}{\epsilon}\mu_1,\ \lambda' = \mu_3 - \frac{1}{\epsilon}\mu_1.$$

Then as λ, resp. λ', is the pull-back via τ of a class $\tilde{\lambda}$ on $\mathbb{P}^N \times \mathbb{P}^N \times \mathbb{P}^N$, it follows that the following diagram commutes

$$\begin{array}{lccc} \tilde{\lambda}_{|W}\cup : & H^1(W, \mathbb{Q}) & \to & H^3(W, \mathbb{Q}) \\ & j_*\tau'^* \downarrow & & j_*\tau'^* \downarrow \\ \lambda\cup : & H^3(X, \mathbb{Q}) & \to & H^5(X, \mathbb{Q}), \end{array}$$

where the vertical maps are injective. We have the similar result for λ'. On the other hand, identifying W with T, we have

$$\tilde{\lambda}_{|W} = M\lambda_1,\ \tilde{\lambda}'_{|W} = M\lambda_2.$$

As the map $j_*\tau'^* : H^3(T, \mathbb{Q}) \to H^5(T, \mathbb{Q})$ is injective, the fact that the map (3.7) has odd rank thus implies that the map (3.8) has odd rank.

∎

Let us now describe another non trivial necessary condition for a real cohomology algebra M to admit a Hodge structure. Let H^{2i+1} be an even dimensional real vector space, and let H^{4i+2} be a real vector space. Let

$$\mu : \bigwedge^2 H^{2i+1} \to H^{4i+2}$$

be a linear map.

Lemma 3.5 *Suppose there are Hodge structures of respective weights $2i + 1$ and $4i + 2$ on H^{2i+1} and H^{4i+2} for which μ is a morphism of Hodge structures. Then there exists a complex vector subspace $W \subset H^{2i+1}_{\mathbb{C}}$ satisfying the properties:*

1. *$rk\, W = \frac{1}{2} rk\, H^{2i+1}_{\mathbb{C}}$.*
2. *$rk\, \mu_{|\bigwedge^2 W} \leq \frac{1}{2} rk\, \mu$.*

Proof. Indeed, let

$$W := F^i H_{\mathbb{C}}^{2i+1} := H_{\mathbb{C}}^{2i+1,0} \oplus \ldots \oplus H_{\mathbb{C}}^{i+1,i}.$$

Then by Hodge symmetry, 1 is satisfied. Furthermore, if μ is a morphism of Hodge structures, we get

$$\mu(\bigwedge^2 F^i H_{\mathbb{C}}^{2i+1}) \subset H_{\mathbb{C}}^{4i+2,0} \oplus \ldots \oplus H_{\mathbb{C}}^{2i+2,2i},$$

and as $Im\,\mu$ is a real sub-Hodge structure of H^{4i+2}, we conclude by Hodge symmetry for $Im\,\mu$ that we have the condition

$$rank\,\mu(\bigwedge^2 F^i H_{\mathbb{C}}^{2i+1}) \leq \frac{1}{2} rank\,\mu. \tag{3.9}$$

■

Coming back to real cohomology algebras, we can apply this lemma to $H^{2i+1} = M^{2i+1}$, where μ is the cup-product map. Moreover, combining this lemma with the results of the previous section, we can also apply it to more general morphisms of Hodge structures of the form:

$$\mu' = \phi \circ \mu : \bigwedge^2 M^{2i+1} \to M^{4i+2} \stackrel{\phi}{\to} M',$$

where M' will be an adequate Hodge structure of even weight and ϕ will be shown to be a morphism of Hodge structures.

In order to show the effectiveness of this restriction on the structure of cohomology algebras, we need the following lemma:

Lemma 3.6 *Let M be a real or rational vector space of rank $2n$, and let*

$$\mu : \bigwedge^2 M \to Q$$

be a generic linear surjective map to a rational or real vector space Q of rank $q = 2q'$ or $q = 2q' + 1$ satisfying the following numerical conditions :

$$q' \leq \frac{n(n-1)}{2},\ (q - q')(\frac{n(n-1)}{2} - q') > n^2. \tag{3.10}$$

Then there is no complex subspace $W \subset M_{\mathbb{C}}$ of rank n, such that

$$rank\,\mu_{|\bigwedge^2 W} \leq q'. \tag{3.11}$$

Proof. We work inside the space $K = Hom\,(\bigwedge^2 M_{\mathbb{C}}, Q_{\mathbb{C}})$ of $\mathbb{C}$-linear maps $\mu : \bigwedge^2 M_{\mathbb{C}} \to Q_{\mathbb{C}}$ and make a dimension count. The dimension of the Grassmannian $Grass(n, M_{\mathbb{C}})$ of n-dimensional subspaces of $M_{\mathbb{C}}$ is n^2. For fixed $W \in Grass(n, M_{\mathbb{C}})$, the codimension of the (Zariski closed) space consisting of $\mu \in K$ satisfying the condition that $rank\,\mu_{|\bigwedge^2 W} \leq q'$ is equal to $(q-q')(\frac{n(n-1)}{2} - q')$ assuming this number is ≥ 0, which is implied by our first assumption. It follows immediately that under our first assumption in (3.10), the codimension of the set of μ for which there exists a W such that (3.11) holds is at least $q'(\frac{n(n-1)}{2} - q') - n^2$. This is positive under our second assumption in (3.10), and it follows that this Zariski closed subset of K is a proper subset. ■

Let us now construct an example of a symplectic compact manifold X whose real cohomology algebra does not satisfy the criterion given by Lemma 3.5.

Example 3.7 *In lemma 3.6, we make $n = 5$, $q' = 5$, $q = 2q' + 1$. We now consider a real torus T of dimension 10, and let K be any simply connected compact Kähler manifold satisfying the condition that rank $H^2(K, \mathbb{Q}) = 11$ and that the cohomology of K is generated in degree 2.*

We consider now $T \times K$, which can be endowed with the structure of a compact Kähler manifold, hence in particular is a compact symplectic manifold. We have, using Künneth decomposition and Poincaré duality, an inclusion

$$Hom\,(H^2(T, \mathbb{Q}), H^2(K, \mathbb{Q})) \subset H^{10}(T \times K, \mathbb{Q}). \tag{3.12}$$

Choose now a generic surjective map

$$\mu : \bigwedge^2 H^1(T, \mathbb{Q}) = H^2(T, \mathbb{Q}) \to H^2(K, \mathbb{Q})$$

and let $\lambda \in H^{10}(T \times K, \mathbb{Q})$ be the class which is the image of μ under the map (3.12).

Our example will be the symplectic manifold $\mathbb{P}(E)$, for any complex vector bundle E on $Y := T \times K$ such that

$$c_1(E) = 0,\ c_5(E) = m\lambda$$

for some non zero integer m.

Let us now combine Theorem 2.8 and Lemma 3.5 to show the following result.

Proposition 3.8 *The compact symplectic manifold $\mathbb{P}(E)$ has the property that there is no Hodge structure on the real cohomology algebra $H^*(\mathbb{P}(E), \mathbb{R})$.*

Proof. Suppose the conclusion is not satisfied. We first use Theorem 2.8 (version with real coefficients) to conclude that under our assumptions, there should be a real Hodge structure on $H^*(Y, \mathbb{R}) = H^*(T \times K, \mathbb{R})$ for which λ is a Hodge class.

We next use Theorem 2.7, (version with real coefficients) to conclude that there are real Hodge structures on $H^*(T, \mathbb{R})$ and $H^*(K, \mathbb{R})$ which induce the Hodge structure on $H^*(T \times K, \mathbb{R})$. The class λ being a Hodge class on $T \times K$, we conclude that the corresponding morphism

$$\mu : \bigwedge^2 H^1(T, \mathbb{R}) \to H^2(K, \mathbb{R})$$

is a morphism of Hodge structures. But this contradicts the fact that μ is generic, so that by Lemma 3.6, the map μ does not satisfy the necessary condition (3.9), with $2i + 1 = 1$ in this case.

∎

Remark 3.9 The two criteria we applied in this section to detect the non-existence of real Hodge structures on the cohomology algebras of certain symplectic manifolds, and the proof of their effectiveness, are not only different geometrically. The difference of nature is made clear in the fact that in the previous case, our criterion did not apply to a generic choice of λ_1, λ_2, and indeed, for a generic choice of λ_0, λ_1, λ_2, there exists a rational Hodge structure on the cohomology of the variety we constructed.

In the second example, a *generic* choice of μ will lead to an example where the criterion applies.

3.2 Constraints coming from the rational Hodge structure

In this section, we want to use Theorem 2.8 to construct a compact symplectic manifold X, whose cohomology algebra does not carry any Hodge structure, polarized or not. Working a little more, one can see that the *rational* information is actually needed, the information given by the real cohomology algebra being too weak.

Here, it is hard to formulate the criterion in a general way. The starting point is however Theorem 2.8, which tells us that for some types of manifolds, Hodge structures on their cohomology algebras come from Hodge structures on a certain subalgebra, for which certain classes must be Hodge classes. This point can be combined with the general observation made in [20] that a rich cohomology algebra may prevent a cohomology algebra with Hodge structure to carry any non trivial Hodge class.

We will content ourselves illustrating the combination of the two arguments on an example.

Example 3.10 *We start with the simplest example Y constructed in [20] of compact Kähler manifold not having the rational cohomology algebra of a projective complex manifold, namely, we consider a complex torus*

$$T = \Gamma_{\mathbb{C}}/(\Gamma^{1,0} \oplus \Gamma),$$

where Γ is a lattice of even rank $2n$; we assume there is an endomorphism ϕ acting on Γ, and that $\Gamma^{1,0} \subset \Gamma_{\mathbb{C}}$ is the eigenspace associated to the choice of n complex eigenvalues of ϕ, not pairwise conjugate. Here ϕ is assumed to have only complex eigenvalues, and to satisfy the following condition :

$$\text{The Galois group of the splitting field of } \mathbb{Q}[\phi] \text{ is the symmetric group in } 2n \text{ letters acting on the } 2n \text{ eigenvalues of } \phi. \tag{3.13}$$

The torus T admits then the endomorphism ϕ_T induced by the $\mathbb{C}$-linear extension of ϕ acting on $\Gamma_{\mathbb{C}}$, which preserves $\Gamma^{1,0}$ and Γ.

The Kähler manifold Y was obtained by successive blow-ups of $T \times T$. Namely, observing that the four subtori

$$T_1 = T \times 0,\ T_2 = 0 \times T,\ T_3 = Diag(T),\ T_4 = Graph(\phi_T)$$

of $T \times T$ meet pairwise transversally in finitely many points $x_1, \ldots, x_N$, we first blow-up the x_i's, getting $\widetilde{T \times T}_{x_1,\ldots,x_N}$; then the proper transforms $\widetilde{T}_i$ are smooth and disjoint, and we get Y by blowing them in $\widetilde{T \times T}_{x_1,\ldots,x_N}$.

We shall denote $\tau : Y \to T \times T$ the natural map, which is the composition of two blow-ups.

Let us choose now any integral cohomology class $\lambda \in H^2(Y, \mathbb{Z})$. Then $\lambda = c_1(L)$, where L is a C^∞ complex line bundle on Y. Our example will be the manifold

$$X = \mathbb{P}(L \oplus L^{-1}).$$

Let us show:

Theorem 3.11 *If* $dim\, T \geq 3$ *and* $0 \neq \lambda^2$ *belongs to* $\tau^* H^4(T \times T, \mathbb{Q}) \subset H^*(Y, \mathbb{Q})$, *then the rational cohomology algebra of the manifold* X *does not admit a Hodge structure.*

Proof. We first observe that the manifold Y has its cohomology generated in degrees 1 and 2. Indeed, this is true for the torus $T \times T$, (for which degree 1 suffices), thus also for $\widetilde{T \times T}_{x_1,\ldots,x_N}$ by adding the classes of exceptional divisors over points. To see that this remains true for Y, observe that the restriction maps in cohomology

$$H^*(\widetilde{T \times T}_{x_1,\ldots,x_N}, \mathbb{Q}) \to H^*(\widetilde{T}_i, \mathbb{Q})$$

are surjective for all i. It thus follows that the cohomology of Y is generated (as an algebra) by the pull-back of the cohomology of $\widetilde{T \times T}_{x_1,\ldots,x_N}$ and the classes e_i of the exceptional divisors over $\widetilde{T}_i$.

We can thus apply Theorem 2.8 and conclude that for any Hodge structure on the rational cohomology algebra $H^*(X, \mathbb{Q})$, the sub-algebra $\pi^* H^*(Y, \mathbb{Q})$ is a sub-Hodge structure, and furthermore, for the induced rational Hodge structure on $H^*(Y, \mathbb{Q})$, the Chern class $c_2(L \oplus -L) = -\lambda^2$ is a Hodge class.

We now briefly recall the analysis made in [20]: We proved that for any rational Hodge structure on $H^*(Y, \mathbb{Q})$, the classes e_i are Hodge classes. Studying the cup-product maps (which must be morphisms of Hodge structures of bidegree $(1,1)$)

$$e_i \cup : H^1(Y, \mathbb{Q}) \to H^3(Y, \mathbb{Q}),$$

where $H^1(Y, \mathbb{Q}) \cong H^1(T \times T, \mathbb{Q})$, we then concluded :

1. The induced (weight 1) Hodge structure on $H^1(T \times T, \mathbb{Q})$ is induced by a Hodge structure on $H^1(T, \mathbb{Q})$, that is, is the direct sum of two copies of a rational Hodge structure on $H^1(T, \mathbb{Q})$.

2. Furthermore this Hodge structure must admit the automorphism ${}^t\phi$. (Observe that $H^1(T, \mathbb{Q}) \cong \Gamma^*_{\mathbb{Q}}$, so that ${}^t\phi$ acts on it.)

The proof is now finished because we know that the class λ^2 must be a Hodge class in $H^4(Y, \mathbb{Q})$, and we made the assumption that it belongs to the sub-Hodge structure $\tau^* H^4(T \times T, \mathbb{Q}) = \bigwedge^4 H^1(Y, \mathbb{Q}) \subset H^4(Y, \mathbb{Q})$. Thus it must be a non trivial Hodge classes in $H^4(T \times T, \mathbb{Q})$, for the Hodge structure induced by the Hodge structure above on $H^1(T, \mathbb{Q})$. However, as in [20] (see [21] for more details of such computations), using assumption (3.13), an easy irreducibility argument for the action of ${}^t\phi$ on certain natural direct summands of $H^*(T, \mathbb{Q}) \otimes H^*(T, \mathbb{Q})$ shows that there is no non zero Hodge classes in $H^4(T \times T, \mathbb{Q})$ for any Hodge structure satisfying the conclusions 1, 2 above.

∎

3.3 Constraints coming from polarizations

We have made essentially no use of the polarization in the previous sections. Furthermore, we used a lot odd dimensional cohomology, for the following reason: a Hodge structure of odd weight always gives a non trivial information. This is not the case for Hodge structures of even weight $2k$. In this case, we can always consider the trivial Hodge structure, for which everything is of type (k,k). Furthermore, if we have a cohomology algebra M with trivial odd degree part, then we can always put the trivial Hodge structure on each term M^{2k}, and this will give us a cohomology algebra with Hodge structure.

In the presence of polarization, and assuming the dimension of M is divisible by 4, we meet now the following restriction given by the Hodge index theorem 3.12. Let us define the signature $\tau(M)$ of a cohomology algebra M of dimension $4m$ as the signature on the intersection form on M^{2m}. This signature is only defined up to sign if we did not fix the isomorphism

$$M^{4m} \cong \mathbb{R}.$$

Theorem 3.12 *Let M be a cohomology algebra of dimension $4m$ which is endowed with a trivial Hodge structure. Then, if this Hodge structure can be polarized, we have*

$$\tau(M) = \pm \sum_i (-1)^i rk\, M^{2i}.$$

Proof. Indeed, if we look at the Riemann second bilinear relations (1.7), they give us the signs on the intersection form on the real part of the $(p,q) \oplus (q,p)$ part of the pieces of the Lefschetz decomposition on the middle piece M^{2n}. (Here, all the signs may be reversed by changing the orientation.) As a result one gets the Hodge index formula [22], 6.3.2, which gives the signature τ as an alternate sum of $m^{p,q}$ numbers, $m^{p,q} := rk\, M^{p,q}$. In our case, the Hodge decomposition is trivial, so we get the formula

$$\tau = \pm \sum_i (-1)^i m_{2i},\, m^{2i} := rk\, M^{2i}. \tag{3.14}$$

■

Let us now use this condition, combined with Deligne's Lemma 1.9, to construct a symplectic compact manifold X such that $M = H^*(X,\mathbb{Q})$ carries a rational Hodge structure, and such that adequate symplectic classes ω on X satisfy the Lefschetz property, but such that M does not carry any *polarized* Hodge structure.

Example 3.13 *We consider a $K3$ surface S, and consider a basis $a_1, \ldots, a_{22}$ of $H^2(S,\mathbb{Q})$ consisting of symplectic classes; more precisely, we assume that a_i is the cohomology class of a symplectic form α_i close to a given symplectic form α, so that $\sum_i \alpha_i$ is again a symplectic form.*

For an adequate integer coefficient $l >> 0$, one knows by [12] that one can construct embeddings $\phi_i : S \to \mathbb{P}^N$ such that $\phi_i^*\Omega = l\alpha_i$, where Ω is the Fubini-Study Kähler form on $\mathbb{P}^N$. Then $\sum_i \phi_i^*\Omega = l\sum_i \alpha_i$ is again a symplectic form on S.

We consider now the embedding $\psi := (\phi_1, \ldots, \phi_{22})$ of S into $(\mathbb{P}^N)^{22}$. With the above choice of ϕ_i's, this provides a symplectic submanifold of $(\mathbb{P}^N)^{22}$ endowed with the product Kähler form.

The symplectic manifold we will consider will be the symplectic blow-up of $(\mathbb{P}^N)^{22}$ along $\psi(S)$. This is a symplectic manifold with symplectic class given by $\tau^*(\sum_i pr_i^*\omega) - \epsilon e$, where $\tau : X \to (\mathbb{P}^N)^{22}$ is the blowing-up map and e is the cohomology class of the exceptional divisor.

Let us show:

Theorem 3.14 *Any Hodge structure on $H^*(X,\mathbb{Q})$ is trivial (that is $H^{2i}(X,\mathbb{Q})$ is completely of type (i,i)). Furthermore the trivial Hodge structure cannot be polarized.*

Proof. We use lemma 1.9 to show that for any Hodge structure on $H^*(X,\mathbb{Q})$, the cohomology $H^2(X,\mathbb{Q})$ must be completely of type $(1,1)$. Indeed, the cohomology of X in degree 2 is generated by the $h_i := \tau^* pr_i^* h$ and by e. We observe that for each $i = 1,\ldots,22$, the line generated by h_i is an irreducible component of the set

$$Z \subset H^2(X,\mathbb{C}),\ Z = \{\alpha \in H^2(X,\mathbb{C}),\ \alpha^{N+1} = 0\}.$$

Thus Lemma 1.9 implies that this line is generated by a Hodge class, as it is defined over $\mathbb{Q}$. Finally, the last generator e must also be of type $(1,1)$, because the $(2,0)\oplus(0,2)$ part has even rank, and we just proved above that it has rank ≤ 1.

Next we observe that the cohomology of our variety X is generated in degree 2. This follows from the computation of the cohomology of a blown-up variety (cf [22], 7.3.3 in the complex case, the symplectic case is computed in the same way), and from the following facts:

1) The variety $(\mathbb{P}^N)^{22}$ satisfies the property that its cohomology is generated in degree 2.

2) The restriction map on cohomology

$$\psi^* : H^*((\mathbb{P}^N)^{22},\mathbb{Q}) \to H^*(S,\mathbb{Q})$$

is surjective.

It follows that if $j : E \hookrightarrow X$ is the inclusion, and $\tau' : E \to S$ is the restriction of τ, then for $\alpha = \psi^*\beta \in H^*(S,\mathbb{Q})$, we have

$$j_*{\tau'}^*\alpha = e \cup \tau^*\beta \text{ in } H^*(X,\mathbb{Q}),$$

and this implies that the cohomology of X is also generated in degree 2.

As the Hodge structure on $H^2(X,\mathbb{Q})$ is trivial, the Hodge structure on all cohomology groups $H^{2i}(X,\mathbb{Q})$ are trivial too. Furthermore, X has no odd dimensional cohomology.

It remains to see why the trivial Hodge structure cannot be polarized. For this we apply Theorem 3.12, which tells us that in this case we should have

$$\tau(X) = \pm \sum_i (-1)^i b_{2i}. \tag{3.15}$$

This formula must give the signature (defined up to sign) of the intersection form on the middle part of any cohomology algebra endowed with a trivial Hodge structure which can be polarized. We can now easily construct a projective variety with the same Betti numbers and which has the property that the Hodge structures on its cohomology are trivial. Namely we start from $S' = \mathbb{P}^2$ blown-up at 21 points, choose a basis $b_1,\ldots,b_{22}$ of $H^2(X',\mathbb{Q})$ consisting of Chern classes of very ample line bundles L_i and imbed holomorphically S' in $(\mathbb{P}^N)^{22}$, using the morphisms ϕ_i' given by the line bundles L_i. Then we consider the projective variety X' defined by blowing-up S' in $(\mathbb{P}^N)^{22}$.

As X' is projective, its cohomology carries a polarized Hodge structure. As the blown-up surface X' has a trivial Hodge structure, the Hodge structure on $H^*(X', \mathbb{Q})$ is trivial. Note also that, as S' has only trivial Hodge structures, so does X', and thus X' has the same Hodge numbers as X. Thus the signature of X' is given by formula (3.15).

Hence, to conclude that the trivial Hodge structure on $H^*(X, \mathbb{Q})$ cannot be polarized, it suffices to show that the absolute value of the signature of X is different from that of X', hence does not satisfy formula (3.15).

This is quite easy, because the middle cohomology $H^{22N}(X, \mathbb{Q})$ is a direct sum

$$H^{22N}(\mathbb{P}^{22N}, \mathbb{Q}) \bigoplus e^{11N}\mathbb{Q} \bigoplus j_*(e^{11N-3}\tau'^* H^4(S, \mathbb{Q})) \bigoplus j_*(e^{11N-2}\tau'^* H^2(S, \mathbb{Q})),$$

where the first term is orthogonal to the three other ones, the second and the third are isotropic and dual, and orthogonal to the last one, and the intersection form of X restricted to the term $j_*(e^{11N-2}\tau'^* H^2(S, \mathbb{Q})$ is equal to the intersection form on $H^2(S, \mathbb{Q})$, with opposite sign.

We do the same computation with X' and we conclude that the difference $\tau(X) - \tau(X')$ is equal to $\tau(S') - \tau(S)$, hence is non zero.

The argument is not quite complete, as we also have to show that we do not have $\tau(X) = -\tau(X')$. This is easily checked for large N.

■

4 Further restrictions

Up to now, we have been studying the topological constraints on compact Kähler manifolds via the constraints imposed to the cohomology algebra by the existence of a (polarized) Hodge structure. In this final section, we want to show that there are in fact other constraints on the cohomology algebra.

Theorem 4.1 *There exists a compact symplectic manifold whose rational cohomology algebra carries a rational polarizable Hodge structure, but is not isomorphic to the rational cohomology algebra of any compact Kähler manifold.*

In fact our example will even be a manifold which does not have the rational cohomology algebra of a compact Kähler manifold, but satisfies the following properties:

1. The cohomology algebra of X admits a polarized rational Hodge structure.
2. X has the real cohomology algebra of a compact Kähler manifold.

Example 4.2 *We consider a* 3*-dimensional torus T, which admits complex multiplication by a number field K, with $[K : \mathbb{Q}] = 6$. This means that K acts on T, hence on $H^1(T, \mathbb{Q})$, and this makes the space $H^1(T, \mathbb{Q})$ a* 1*-dimensional K-vector space.*

Let k_i, $i = 1, \ldots, 6$ be a basis of K over $\mathbb{Q}$ and for each i, let

$$\gamma_i \in Hom\,(H^1(T, \mathbb{Q}), H^1(T, \mathbb{Q})) \subset H^6(T \times T, \mathbb{Q})$$

be given by the action of k_i on $H^1(T, \mathbb{Q})$. Let also f_1, $f_2 \in H^6(T \times T, \mathbb{Q})$ be the respective classes of the fibers of the projections $pr_1 : T \times T \to T$, $pr_2 : T \times T \to T$.

Note that we have

$$H^6(T\times T,\mathbb{Q})\otimes H^6(T\times T,\mathbb{Q})\subset H^{12}(T\times T\times T\times T,\mathbb{Q}) \tag{4.16}$$

by Künneth decomposition, and

$$H^6(T\times T,\mathbb{Q})\otimes H^6(T\times T,\mathbb{Q})\cong Hom\,(H^6(T\times T,\mathbb{Q}),H^6(T\times T,\mathbb{Q})) \tag{4.17}$$

by Poincaré duality on $T\times T$.

Let $V\subset H^6(T\times T,\mathbb{Q})$ be the subspace generated by $f_1,f_2,\gamma_1,\ldots,\gamma_6$. Observing that the intersection pairing is non degenerate on V, let

$$P: H^6(T\times T,\mathbb{Q})\to H^6(T\times T,\mathbb{Q})$$

denote the orthogonal projector onto V.

By (4.16) et (4.17) P can be seen as a rational cohomology class of degree 12 *on $Y:=T^4$. Furthermore, P is a Hodge class, because it corresponds to an endomorphism of Hodge structure (an orthogonal projection onto a sub-Hodge structure) acting on $H^*(T\times T,\mathbb{Q})$.*

Let G be a complex vector bundle of rank r on Y such that $c_i(G)=0$ for $i\neq 0$, 6*, and $c_6(G)$ is a non zero multiple of P, say $c_6(G)=mP$.*

Similarly the classes e, f of the fibers of the projections pr_1, resp. pr_2 from $Y=T^4=T^2\times T^2$ to T^2 are degree 12 *Hodge classes on Y. Let E, F be complex vector bundles of rank s on Y with the property $c_i(E)=c_i(F)=0$, $i\neq 0$, 6, $c_6(E)=m'e$, $c_6(F)=m'f$ for some non zero integer m'.*

We define X to be the fibered product over Y of the projective bundles $\mathbb{P}(E)$, $\mathbb{P}(F)$, $\mathbb{P}(G)$:

$$X:=\mathbb{P}(E)\times_Y\mathbb{P}(F)\times_Y\mathbb{P}(G).$$

Let us prove:

Theorem 4.3 *1. The cohomology algebra $H^*(X,\mathbb{Q})$ is endowed with a natural Hodge structure.*

2. The real cohomology algebra $H^*(X,\mathbb{R})$ does not depend on the choice of number field K satisfying the condition that $K\otimes\mathbb{R}\cong\mathbb{C}^3$.

3. For certain choices of K, and adequate choices of r, s, m, m', the variety X is projective (in particular Kähler). With the same m, m', r, s, for a "generic" choice of K, X does not have the rational cohomology algebra of a Kähler manifold.

Proof. Statement 1 follows from the explicit computation of the cohomology algebra of X, which is the fibered product of the projective bundles $\mathbb{P}(E)$, $\mathbb{P}(F)$, $\mathbb{P}(G)$ over $Y=T^4$. This cohomology algebra is then generated over $H^*(T^4,\mathbb{Q})$ by $h_E:=c_1(\mathcal{L}_{\mathbb{P}(E)})$, where $\mathcal{L}_{\mathbb{P}(E)}$ is the dual of the relative Hopf line bundle, and similarly h_F and h_G, with the relations

$$h_E^s=-\pi^*c_6(E)h_E^{s-6},\ \ h_F^s=-\pi^*c_6(F)h_F^{s-6},\ \ h_G^r=-\pi^*c_6(G)h_G^{r-6},$$

where $\pi: X\to Y$ is the structural map. This presentation is due to the fact that $c_i(E)=0$, $i\neq 0$, 6 and similarly for F and G.

As we know that $c_6(E)$, $c_6(F)$, $c_6(G)$ are Hodge classes on Y, there is a Hodge structure on $H^*(X,\mathbb{Q})$, inducing the Hodge structure on $H^*(Y,\mathbb{Q})$ and obtained by declaring h_E, h_F and h_G to be of type $(1,1)$.

For the proof of 2, observe that the real cohomology algebra $H^*(X,\mathbb{R})$ depends only on the representation of the algebra $K\otimes_{\mathbb{Q}}\mathbb{R}$ on $H^1(T,\mathbb{R})$. Thus we conclude that the $\mathbb{R}$-algebra $H^*(X,\mathbb{R})$ does not depend on the choice of field K satisfying the condition that $K\otimes_{\mathbb{Q}}\mathbb{R}=\mathbb{C}^3$.

It remains to prove 3.

First of all, we note that we can choose K in such a way that T and thus $Y=T^4$ are abelian varieties. The classes e, f, f_1, f_2, γ_i are all classes of algebraic cycles. Here we use the fact that γ_i are algebraic, which is a general fact (see [17]): the Künneth component of type $(n-1,1)$ of a codimension n algebraic cycle class in a product of smooth projective varieties Z, Z', $dim\, Z=n$, is again an algebraic cycle class.

As Y is projective, we know that the Chern character

$$ch: K_{0,alg}\otimes\mathbb{Q}\to CH(Y)\otimes\mathbb{Q}$$

is an isomorphism, and this implies that for adequate choices of ranks r, s and integers m, m' there exist *algebraic* vector bundles E, F of rank s, G of rank r on Y satisfying the conditions :

$$c_i(E)=c_i(F)=c_i(G)=0,\ i\neq 6,\ c_6(E)=m'e,\ c_6(F)=m'f,\ c_6(G)=mP.$$

The corresponding manifold

$$X=\mathbb{P}(E)\times_Y\mathbb{P}(F)\times_Y\mathbb{P}(G)$$

is then projective, which proves the first statement.

Let us now show that for a number field K satisfying the condition (3.13), that is, its Galois group is as large as possible, the rational cohomology algebra of the symplectic manifold X (independently of the integers r, s, m, m') is not isomorphic to the rational cohomology algebra of a compact Kähler manifold. The key point here, as in [20] or in Theorem 3.11 above, is that in this case, the presence of such an algebra acting by isogenies on the complex torus Y prevents it to be an abelian variety. In turn, this will prevent the Hodge classes used below to come from Chern classes of reflexive analytic coherent sheaves, as in [23].

So assume to the contrary that there exists a compact Kähler manifold Z which has its cohomology algebra isomorphic to that of X. Let $\pi': Z\to Alb\, Z$ be the Albanese map of Z. Topologically, π' induces the isomorphism

$$\pi'^*: H^1(Alb\, Z,\mathbb{Q})\cong H^1(Z,\mathbb{Q})$$

and the map induced by cup-product on Z:

$$\pi'^*: H^l(Alb\, Z,\mathbb{Q})\cong\bigwedge^l H^1(Z,\mathbb{Q})\to H^l(Z,\mathbb{Q}).$$

As the cohomology algebra of X and Z are isomorphic, we conclude that the map π'^* is injective in top degree for Z, as it is the case for X, and this implies that

$$\pi': Z\to Alb\, Z$$

is surjective.

Observe also for future use that, via the isomorphism of cohomology algebras

$$H^*(X,\mathbb{Q}) \cong H^*(Z,\mathbb{Q})$$

the morphism π'^* identifies to $\pi^*: H^*(Y,\mathbb{Q}) \to H^*(X,\mathbb{Q})$.

Next, we have the following lemma:

Lemma 4.4 *The torus $Alb\, Z$ must be isogenous to a product $T_1^2 \times T_2^2$ where T_1 and T_2 are three-dimensional tori on which the field K acts by isogenies.*

We postpone the proof of this lemma and conclude the proof of Theorem 4.3 as follows: As the torus $Alb\, Z$ is of the form $T_1^2 \times T_2^2$, where the T_i's are 3-dimensional tori with an action of K, and as K satisfies condition (3.13), one proves as in [21] that

$$Hdg^2(Alb\, Z) = 0,\ Hdg^4(Alb\, Z) = 0.$$

This torus thus satisfies the condition of the Appendix of [23], and we thus conclude as in [23] that any reflexive analytic coherent sheaf $\mathcal{F}$ on $Alb\, Z$ is a vector bundle with trivial Chern classes. In particular, by the Riemann-Roch formula for complex vector bundles on compact complex manifolds (see [1]), any reflexive coherent sheaf on $Alb\, Z$ satisfies

$$\chi(Alb\, Z, \mathcal{F}) = 0. \tag{4.18}$$

Next we observe that the morphism $\pi': Z \to Alb\, Z$ is projective, that is, there exist line bundles on Z which are relatively ample w.r.t. π'. Indeed, this follows from iterated applications of (the proof of) Theorem 2.8, which show that for any Hodge structure on the rational cohomology algebra $H^*(Z,\mathbb{Q})$, the sub-algebra $Im\, \pi'^*$ (which identifies as mentioned above to $Im\, \pi^*$) is a sub-Hodge structure, and that $H^2(Z,\mathbb{Q})$ is generated by $\pi'^*H^2(Alb\, Z,\mathbb{Q})$ and by three Hodge classes of degree 2 (corresponding to h_E, h_F, h_G). It follows that for any fiber Z_a of π'^*, the image of the map $H^2(Z,\mathbb{Q}) \to H^2(Z_a,\mathbb{Q})$ is of type $(1,1)$. On the other hand, this image contains a Kähler class on Z_a. We then easily conclude that some rational combinations of the classes h_E, h_F, h_G (transported to Z) restricts to a Kähler class on any fiber Z_a, which implies by Kodaira theorem that this rational combination is ample on the fibers of π'.

The contradiction now comes from the following: let $\mathcal{L}$ be a relatively ample line bundle on Z, whose first Chern class $c_1(\mathcal{L})$ is a rational combination of h_E, h_F, h_G. Observe that we can assume that the top self-intersection $c_1(\mathcal{L})^N$, $N = dim\, Z = 12 + r - 1 + 2(s-1)$ is not equal to 0. Indeed, this follows from the relations

$$h_G^r = c_6(G)h_G^{r-6},\ \ h_G^{r+11} = c_6(G)h_G^{r+5} = c_6(G)^2 h_G^{r-1},$$

$$h_G^{r+11}h_E^{s-1}h_F^{s-1} = c_6(G)^2 h_E^{s-1}h_F^{s-1}h_G^{r-1} \neq 0,$$

where the last equality follows from $c_6(G)^2 = m^2P^2 \neq 0$.

As $c_1(\mathcal{L})^N \neq 0$, the Hilbert polynomial $P_{\mathcal{L}}$ defined by $P_{\mathcal{L}}(n) = \chi(Z, \mathcal{L}^{\otimes n})$ is not identically equal to 0. As $\mathcal{L}$ is relatively ample, we have

$$R^i\pi'_*\mathcal{L}^{\otimes n} = 0$$

for large n and $i > 0$, and thus

$$\chi(Z, \mathcal{L}^{\otimes n}) = \chi(Alb\, Z, R^0 {\pi'}^* \mathcal{L}^{\otimes n}).$$

But the coherent sheaf $\mathcal{F}_n = R^0\pi'_*\mathcal{L}^{\otimes n}$ is reflexive on $Alb\, Z$, because the analysis of $H^2(Z)$ shows that no divisor of Z can be contracted to a codimension 2 analytic subset of $Alb\, Z$. Hence by (4.18), $\mathcal{F}_n$ satisfies $\chi(Alb\, Z, \mathcal{F}_n) = 0$, which is a contradiction.

∎

Proof of Lemma 4.4. We use first of all in an iterated way Theorem 2.8, which implies that any Hodge structure on $H^*(X, \mathbb{Q}) \cong H^*(Z, \mathbb{Q})$ comes from a Hodge structure on $H^*(Y, \mathbb{Q})$ for which the classes $c_6(E)$, $c_6(F)$, $c_6(G)$ must be Hodge classes. As we have already seen, $H^*(Y, \mathbb{Q})$ identifies to $H^*(Alb\, Z, \mathbb{Q})$ under the isomorphism

$$H^*(X, \mathbb{Q}) \cong H^*(Z, \mathbb{Q}).$$

Thus the complex torus $Alb\, Z$ satisfies the property that there is an isomorphism of exterior algebras

$$H^*(T^4, \mathbb{Q}) \cong H^*(Alb\, Z, \mathbb{Q})$$

sending the classes e, f and P to Hodge classes.

Recall that $Y \cong T^2 \times T^2$ and that e and f are the fibers of the two projections on T^2. It follows that

$$Ker\,(e\wedge : H^1(Y, \mathbb{Q}) \to H^1(Y, \mathbb{Q})),$$

which is equal to $pr_1^* H^1(T^2, \mathbb{Q})$ is sent to a sub-Hodge structure of $H^1(Alb\, Z, \mathbb{Q})$. Similarly, $Ker\,(f\wedge : H^1(Y, \mathbb{Q}) \to H^1(Y, \mathbb{Q}))$, which is equal to $pr_2^* H^1(T^2, \mathbb{Q})$ is sent to a sub-Hodge structure of $H^1(Alb\, Z, \mathbb{Q})$. Thus we conclude that $Alb\, Z$ is isogenous to a direct sum $T' \oplus T''$ of two tori, in such a way that the isomorphism

$$H^1(T^2 \times T^2, \mathbb{Q}) \to H^1(Alb\, Z, \mathbb{Q})$$

sends $pr_1^* H^1(T^2, \mathbb{Q})$ to $H^1(T', \mathbb{Q})$ and $pr_1^* H^1(T^2, \mathbb{Q})$ to $H^1(T'', \mathbb{Q})$.

We now consider the image of the class P. This is now a Hodge class in $H^*(T' \times T'', \mathbb{Q})$ which lies in $Hom\,(H^*(T', \mathbb{Q}), H^*(T'', \mathbb{Q}))$. Thus its image in $H^*(T'', \mathbb{Q})$ and the orthogonal of its kernel in $H^*(T', \mathbb{Q})$, which both identify to $V \subset H^*(T^2, \mathbb{Q})$, are sub-Hodge structure of $H^*(T'', \mathbb{Q})$ and $H^*(T', \mathbb{Q})$ respectively. Thus it suffices to prove that if a 6-dimensional torus T' admits an isomorphism of cohomology algebras

$$H^*(T \times T, \mathbb{Q}) \cong H^*(T', \mathbb{Q})$$

sending V to a sub-Hodge structure of $H^*(T', \mathbb{Q})$, then T' is of the form T_1^2, where T_1 admits an action of K by isogenies.

We first observe that the complex lines generated by f_1 and f_2 are irreducible components of the set of reducible elements in $V_{\mathbb{C}} \subset \bigwedge^6 H^1(T', \mathbb{C}) = H^6(T', \mathbb{C})$. As these two lines are defined over $\mathbb{Q}$, they must be generated by a Hodge class on T', by lemma 1.9.

We use now the classes f_1, f_2 to show that T' is isogenous to a product

$$T' \cong T'_1 \oplus T'_2$$

of two 3-dimensional complex tori. Indeed, we recover the sub-Hodge structures $H^1(T_i',\mathbb{Q}) \subset H^1(T',\mathbb{Q})$ as the kernel of the morphism of Hodge structures

$$\cup f_i : H^1(T',\mathbb{Q}) \to H^7(T',\mathbb{Q}).$$

Having this decomposition, we look at the image of V in $Hom\,(H^1(T_1',\mathbb{Q}), H^1(T_2',\mathbb{Q}))$ via the natural projection (given by Künneth decomposition and Poincaré duality)

$$H^6(T_1' \times T_2', \mathbb{Q}) \to Hom\,(H^1(T_1',\mathbb{Q}), H^1(T_2',\mathbb{Q})).$$

The image $\overline{V}$ of V under this map is a sub-Hodge structure of rank 6, and the subalgebra L of $End\,H^1(T_1',\mathbb{Q})$ generated by the $\gamma_i^{-1}\circ\gamma_j$ is a sub-Hodge structure of $End\,H^1(T_1',\mathbb{Q})$ which is isomorphic to K as a $\mathbb{Q}$-algebra. As $K\otimes\mathbb{C}$ has no nilpotent element, it follows that $L_{\mathbb{C}}$ has no $(-1,1)$-part in its Hodge decomposition, because the $(-1,1)$-part of the Hodge structure on $End\,H^1(T_1',\mathbb{Q})$ is equal to $Hom\,(H^{0,1}(T_1'), H^{1,0}(T_1'))$ and this is nilpotent.

Thus $L_{\mathbb{C}}$ is purely of type $(0,0)$, and L consists of endomorphisms of T_1'. As L is isomorphic to K, this shows that T_1' has complex multiplication by K. Finally, as $L_{\mathbb{C}}$ is of type $(0,0)$, the same is true of $\overline{V}$, which implies that T_1' and T_2' are isogenous. This concludes the proof of the Lemma. ■

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